An Invitation to Geometry: Image Analysis, Geometric Analysis, and High-dimensional Geometry - part II

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Outline

Review of tutorial schedule:

Part 1 The first 3 lectures were a slightly modified version of the short course I gave at the Institute for Pure and Applied Mathematics (IPAM) summer 2005.

- **Lecture 1.1** Introduction to metrics and regularization: basic concepts and connection to statistical modeling
- **Lecture 1.2** Metrics: examples, data fidelity terms, warping, and face recognition
- **Lecture 1.3** Regularization: examples, denoising and total variation based methods, and geometric analysis

Part 2 In these last two lectures, we will look in a bit more detail at geometric analysis and high dimensional geometry. Given the size of the fields, this will merely suffice to give you a brief look at a few aspects I find useful and interesting.

- **Lecture 2.1** BV funtions and the TV seminorm: a path into geometric analysis.
- **Lecture 2.2** High Dimensional Geometry: Concentration of Measure and a wee bit of Johnson-Lindenstrauss.

Lecture 4: Geometric Analysis Via BV Functions

In the third lecture my stated perspective and goals were:

Perspective: geometric/analytic insights provide the power needed for creating and understanding the best image analysis methods.

My Goal: to motivate you to learn more.

In this fourth lecture, I will look at more closely at geometric analysis, with similar objectives:

Perspective: geometric analysis is fascinating in its own right and the theory of the space of BV functions provides a very nice path into the subject.

My Goal: Again, it is to motivate you to learn more.

Beginning with History: Geometric Measure theory

1960-1961 was a landmark time for geometric measure theory: Three seminal papers appeared.

Federer and Fleming "Normal and Integral Currents" in which they introduced *k-dimensional currents*, the elements of the dual space to the space of smooth k-forms in $\mathbb{R}^n$ that can be seen as k-rectifiable sets with densities (multiplicities) and an orientation. The paper appeared in 1960.

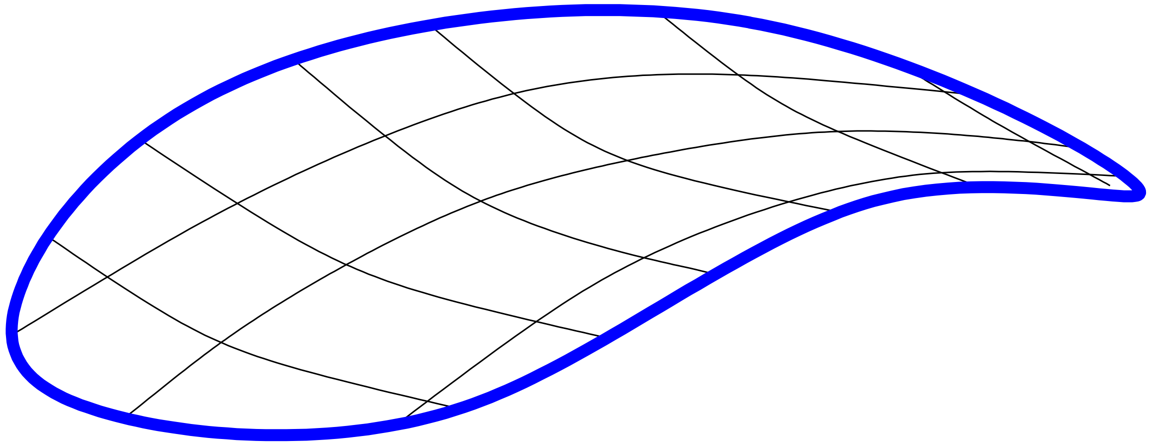
De Giorgi A paper (in Italian) including almost everywhere regularity for area minimizing hypersurfaces. My rough translation of the paper's title is "Oriented Boundaries of minimal measure". He used sets of finite perimeter for this work. The actual date of publication was 1961.

Reifenberg "Solution of the Plateau problem for m dimensional surfaces of varying topological type." Almost everywhere regularity for area-minimizing surfaces of arbitrary codimension. (codimension of A = dimension of space - dimension of A). The paper appeared in 1960.

Beginning with History: Geometric Measure theory

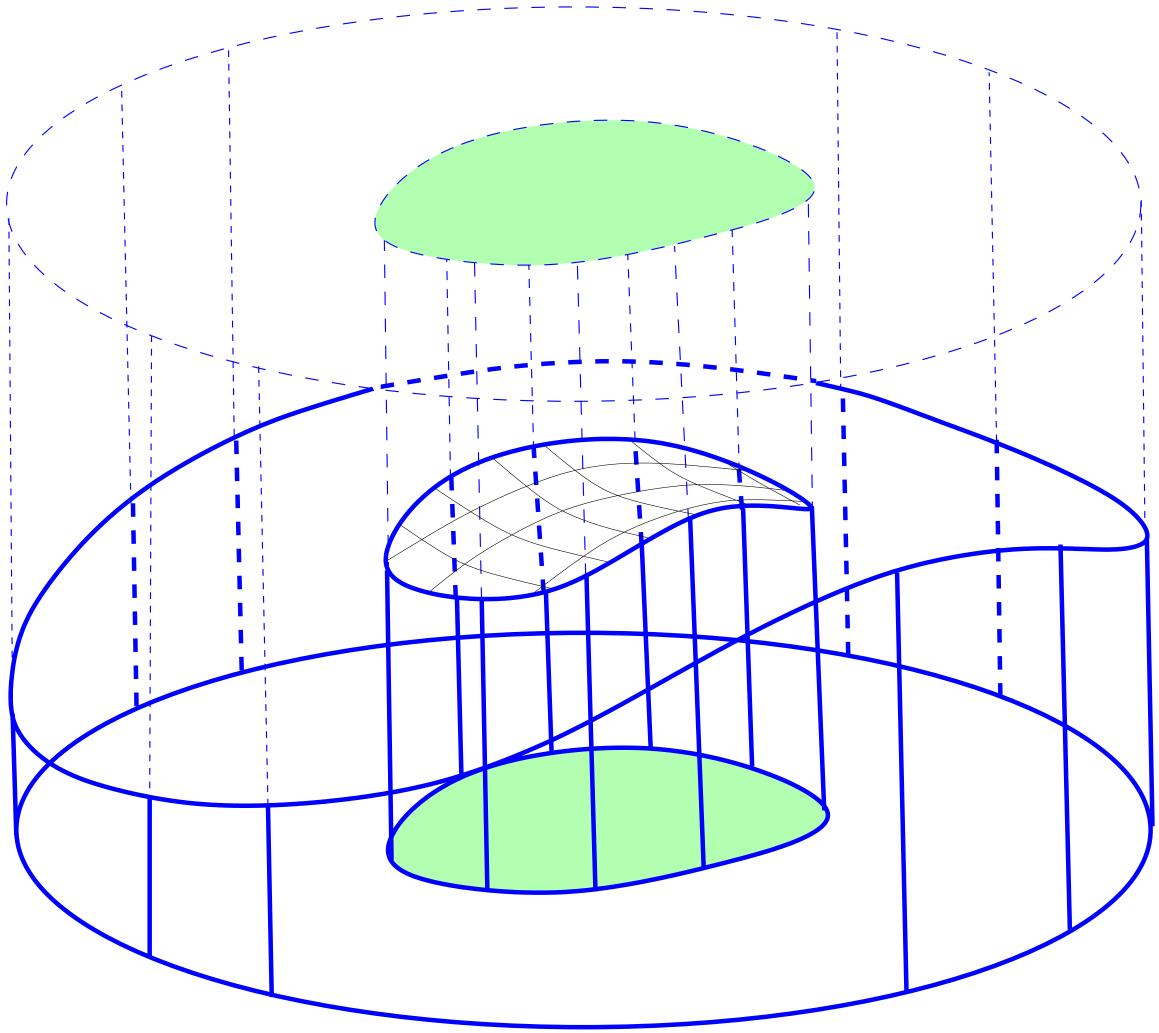
The Problem these papers were all solving was the existence and regularity of minimal surfaces spanning a prescribed boundary.

In the figure below, we find a surface of minimal area spanning the blue boundary.



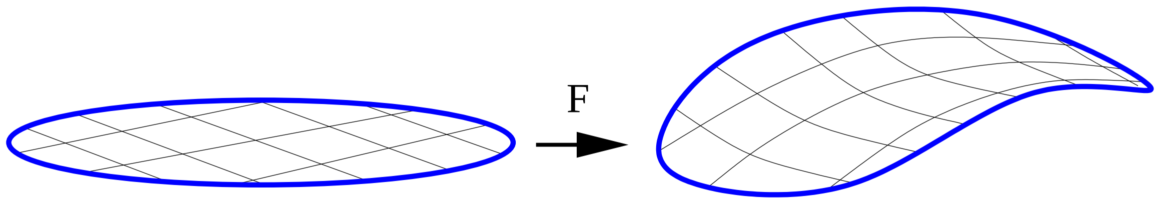
Beginning with History: Geometric Measure theory

One approach: sets with minimizing boundaries.



Beginning with History: Geometric Measure theory

Another approach is through *mapping*: what mapping from the unit disk to $\mathbb{R}^3$, such that the boundary of the disck maps to the boundary in question, minimizes the area of the image of the map?



We require that $F : \partial D \rightarrow B$ where B is the blue boundary. Given this condition is satisfied, we seek the mapping F such that area$(F(D))$ is minimized.

Beginning with History: Geometric Measure theory

Interesting trivia:

* "Geometric Measure Theory" was coined by Herbert Federer who would have chosen "Geometric Integration Theory" if Hassler Whitney had not already published a book by that title.
* One paper by Fred Almgren was both very famous and unpublished during his lifetime. It circulated as a 1700 page(!) mimeographed tome. It was recently published as a 955 page book. Bill Allard told me the paper was truely magnificent: it contained 3 revolutionary new ideas, compared with for example, Nash's embedding paper which introduced 1 new idea.
* When Bill Allard sent his famous 1972 paper on varifolds into the Annals, they eventually sent it back to him to referee! He did, made all the corrections needed (he is very, very careful) and they published it.
* GMT was used by J. Cahn and J. Talyor to predict material features that were then found.
* The classical GMT period was from about 1900 to 1960. Important contributers include Besicovitch, Federer, Morse, Young, De Giorgi, Fleming, and Marstrand.

Beginning with History: Geometric Measure theory

Reference: A very nice paper that I used, in conjunction with conversations with Bill Allard, to get a history and overview of GMT is the "Questions and Answers about Area-Minimizing Surfaces and Geometric Measure Theory" by Fred Almgren (Proceedings of Symposia in Pure Mathematics, Volume **54** (1993), part 1). This paper is included in the volume selected papers of Almgren's referenced at the end of these lectures.

Another nice paper, full of information about the modern era of geometric measure theory is Brian White's paper "The Mathematics of Fred Almgren, Jr." also in the volume edited by Jean Taylor.

BV and TV: mostly review

What is the TV seminorm?

Answer:

$$TV(u,\Omega) \equiv \int_\Omega |\nabla u| dx \tag{1}$$

The dependence on Ω is suppresed if doing so will not lead to confusion.

$BV(\Omega)$ u such that $TV(u) < \infty$ and $\int_\Omega |u| dx < \infty$ (i.e. u is in $L^1(\Omega)$).

BV and TV: mostly review

What functions permit the calculation of TV(u)?

* ∇u has to make sense. $u \in C^1$?
* We generalize by permiting weak derivatives. $u \in W^{1,1}$?
* We only require weak derivatives be a measure. ∇u a Radon measure!
* This turns out to be the most general choice: See Gauss-Green below.

If $u \in C^1(\Omega)$ *total variation of* $u = \int |\nabla u| dx$

If $u \in W^{1,1}$

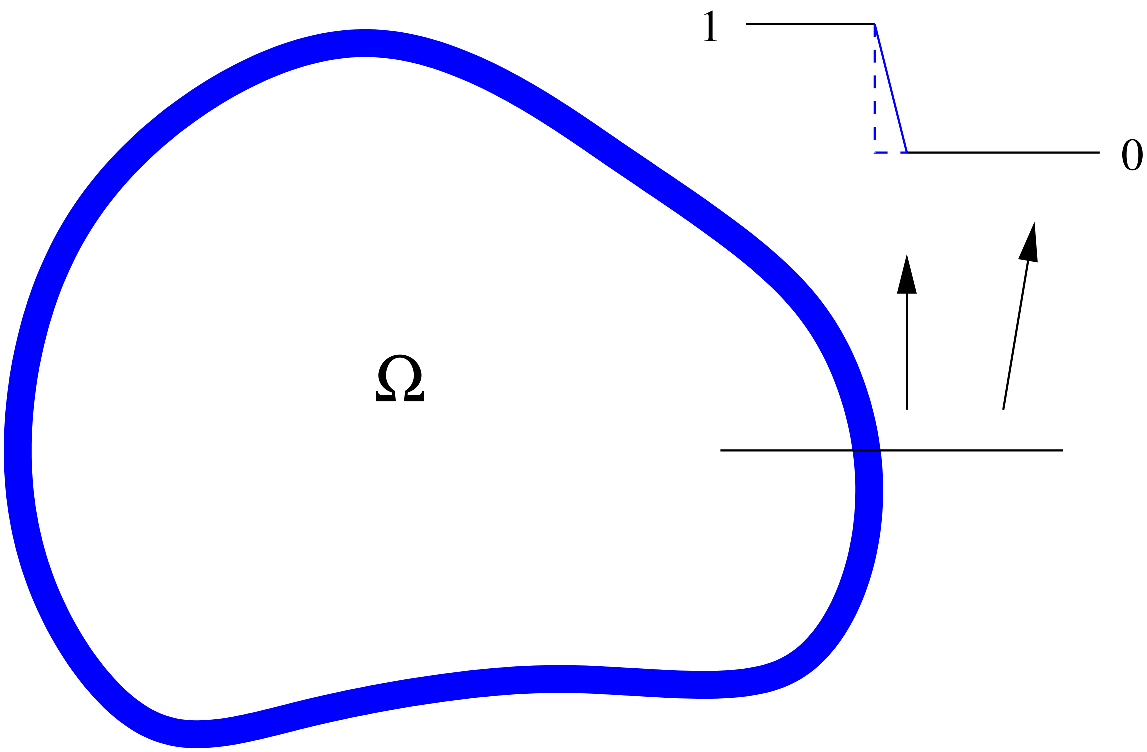
$$\begin{aligned} \int |\nabla u| &= \sup\{\textstyle\int \nabla u \cdot \vec{g} dx \text{ for } |\vec{g}| < 1, \vec{g} \in C_0^1(\Omega; \mathbb{R}^n)\} && (2) \\ &= \sup\{\textstyle\int u \operatorname{div} \vec{g} dx \text{ for } |\vec{g}| < 1, \vec{g} \in C_0^1(\Omega; \mathbb{R}^n)\} && (3) \end{aligned}$$

makes sense.

Finally for $u \in L^1(\Omega)$, we use the last equation to *define* $\int |\nabla u| dx$

BV and TV: mostly review

What is $TV(\chi_\Omega)$ where χ_Ω is the characteristic function of the set Ω?

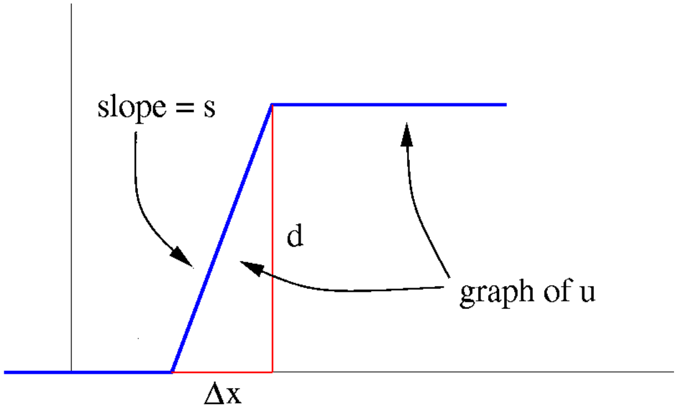


The figure shows an approximate characteristic function; $TV(\chi_\Omega)$ is simply the length of the boundary of the set Ω.

BV and TV: mostly review

What is special about TV(u)? Answer: Discontinuities are kosher:

Consider $F(u) \equiv \int |\nabla u|^p dx$



(calculate) $F(u) = s^p(\Delta x) = \frac{(s\Delta x)^p}{(\Delta x)^{p-1}} = d^p(\Delta x)^{1-p}$

($p > 1$) $F(u) \underset{\Delta x \to 0}{\longrightarrow} \infty$ discontinuities are avoided: smooth u preferred,

($p < 1$) $F(u) \underset{\Delta x \to 0}{\longrightarrow} 0$ discontinuities cost nothing: step u preferred,

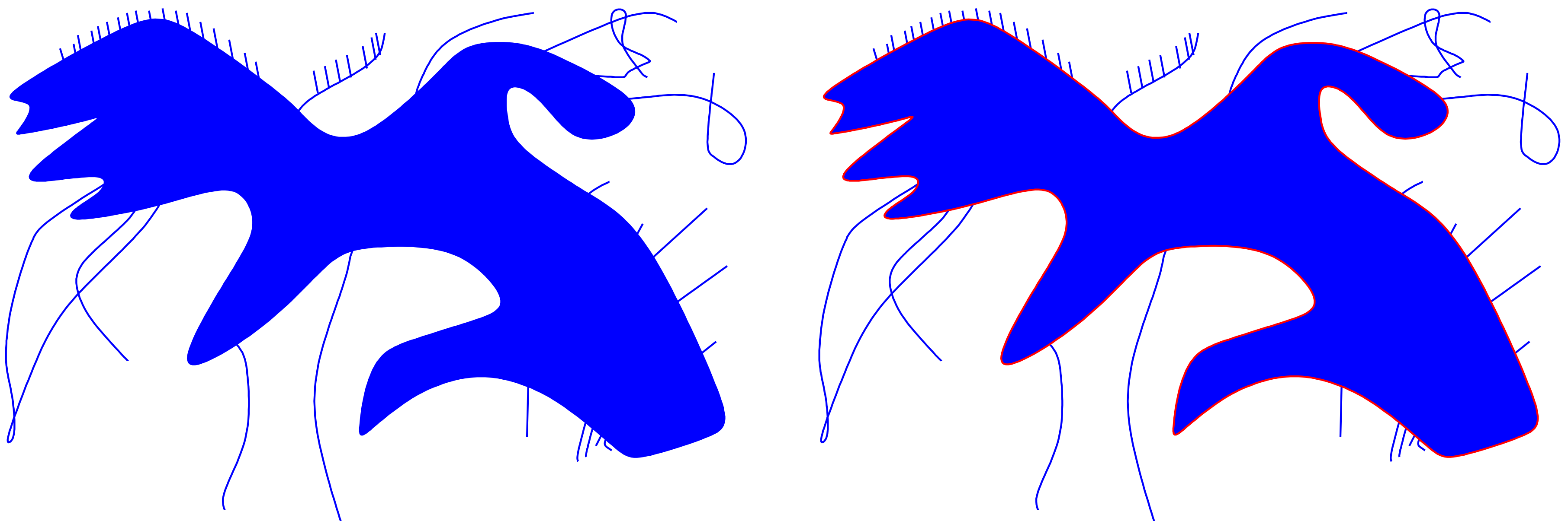
($p = 1$) $F(u) = d$ only jump magnitude "counts", no bias towards smooth or step.

BV and TV: mostly review

What is special about TV(u)?

Answer: "Unimportant" parts of the boundary are ignored:

There is a set $\partial^*\Omega$ called *reduced boundary* of Ω that almost coincides with the measure theoretic boundary $\partial_*\Omega$ that test functions can see. (Almost means $\mathcal{H}^{n-1}$almost everywhere) TV(χ_Ω) picks up the boundary that integration against smooth test functions "sees".



What is special about TV(u)?

Answer: Sets of finite perimeter are the most general sets for which the Gauss-Green theorem holds:

$$\int_\Omega \nabla \cdot \vec{\phi} dx = \int_{\partial_* \Omega} \vec{\phi} \cdot \nu_\Omega d\mathcal{H}^{n-1} \tag{4}$$

BV structure theorem

Radon Measures: general (but nice) measures

Define: $\mu_f(A) \equiv \int_A f dx$ for any positive f in L^1.Then: $f dx = d\mu_f$
Roughly speaking, generalizing f gets us Radon measures.

Another view: BV functions

Typical gradient fields: $d\vec{F}(x) = \vec{\sigma}(x) f(x)$, $\vec{\sigma}(x)$ a unit vector field, $f(x)$ the magnitude $|d\vec{F}(x)|$. Define: the gradient measure $\mu_{dF}(A) \equiv \int_A \vec{\sigma}(x) f(x) dx$.

A BV function is one whose gradient measure can be written $\int_A \vec{\sigma}(x) d\mu$ where μ is a Radon measure.

Rigorously: If $u \in BV(\Omega)$, there is a Radon measure μ and a μ-measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that

I $|\sigma(x)| = 1$ a.e. μ

II $\int_\Omega u \nabla \cdot \phi dx = -\int_\Omega \phi \cdot \sigma d\mu$

for all $\phi \in C_c^1(\Omega; \mathbb{R}^n)$

Approximation properties

How well can we approximate BV functions using smooth functions?

Answer: it depends on what you mean by “approximate”. Here are two theorems.

Theorem 1. [local approximation by smooth functions] *For $f \in BV(\Omega)$ there exists C^∞ functions $\{f_k\}_{k=1}^{\infty}$ such that:*

(1) $f_k \to f$ in $L^1(\Omega)$ and
(2) $||\nabla f_k|| \to ||\nabla f||$ as $k \to \infty$

Theorem 2. [Weak approximation of derivatives] *For each f_k above define*

$$\mu_k(B) \equiv \int_{B\cap\Omega} \nabla f_k dx \tag{5}$$

and

$$\mu(B) \equiv \int_{B\cap\Omega} d[\nabla f]. \tag{6}$$

Then

$$\mu_k \rightharpoonup \mu \tag{7}$$

where $\mu_k \rightharpoonup \mu$ *means* $\int_\Omega f d\mu_k \to \int_\Omega f d\mu$ *for all* f *in* $C_c(\Omega)$*.*

Finally, Whitney's extension theorem permits us to get:

Theorem 3. *Let* $f \in BV(\mathbb{R}^n)$*. Then for every* $\epsilon > 0$ *there is a* C^1 *function* $\bar{f}$ *such that:*

$$\mathcal{L}^n\{x | f(x) \neq \bar{f}(x) \text{ or } \nabla f(x) \neq \nabla \bar{f}(x)\} \leq \epsilon \tag{8}$$

Reduced Boundary and Another Structure Theorem

Suppose $u = \chi_E$ where $E \subset \mathbb{R}^n$. What can we say about the $\nabla \chi_E$?

It should be clear by now that the $\nabla \chi_E$ is a Radon measure concentrated on the topological boundary of E. But we can say a great deal more.

Definition 1. [reduced boundary] *If E is a set of locally finite perimeter in $\mathbb{R}^n$ then we say that $x \in \mathbb{R}^n$ is in the reduced boundary, $\partial^* E$, if*

(1) $||\nabla \chi_E||(B(x,r)) > 0$ *for all* $r > 0$,

(2) $\lim_{r \to 0} \frac{\int_{B(x,r)} \nu_E d||\nabla E||}{\mu(B(x,r))} = \nu_E(x)$, *and*

(3) $|\nu_E(x)| = 1$.

Remark 1. [reduced boundary is "everything"?] *By the Lebesgue-Besicovitch differentiation theorem we have* $||\nabla E||(\mathbb{R}^n - \partial^* E) = 0$. *Remaining questions include, for example,* $\mathcal{H}^{n-1}(\partial E - \partial^* E)$*?*

Reduced Boundary and Another Structure Theorem

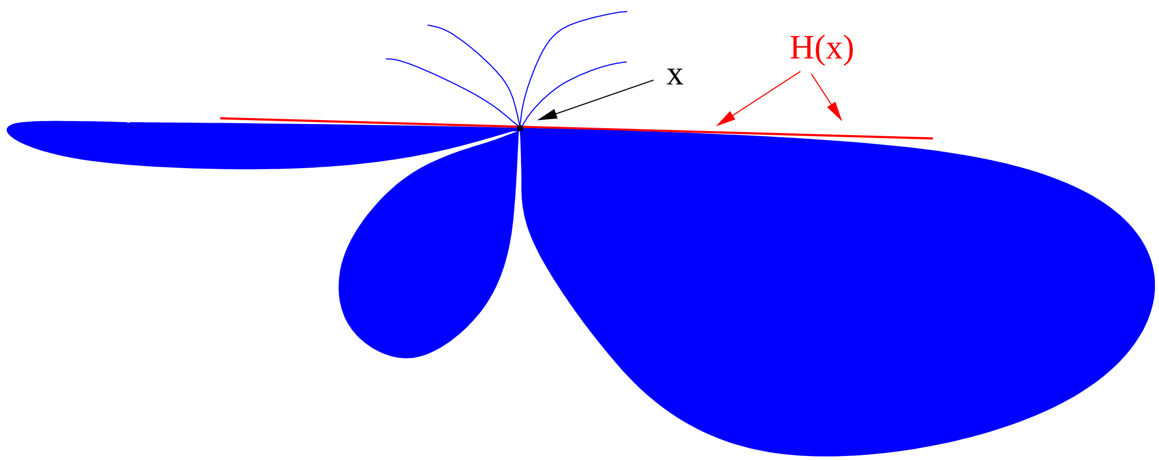
Definition 2. [Approximate Tangent Planes, Half-Spaces] *For* $x \in \partial^* E$, *define*

$$H(x) \equiv \{y \in \mathbb{R}^n | \nu_E(x) \cdot (y - x) = 0\}, \quad (9)$$

$$H^+(x) \equiv \{y \in \mathbb{R}^n | \nu_E(x) \cdot (y - x) \geq 0\}, \quad (10)$$

$$H^-(x) \equiv \{y \in \mathbb{R}^n | \nu_E(x) \cdot (y - x) \leq 0\}, \quad (11)$$

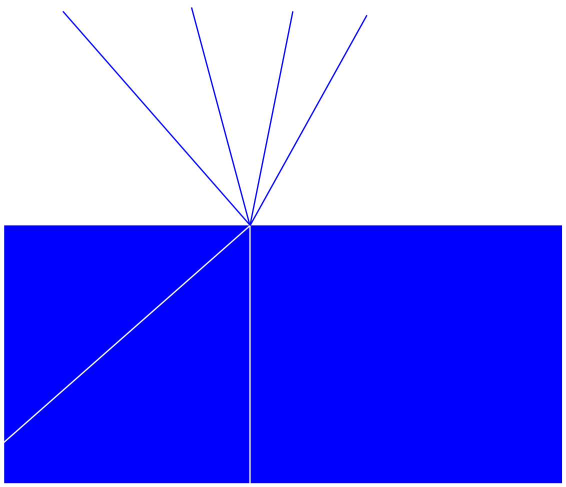
which are, respectively, the approximate tangent hyperplane and the corresponding outside and inside closed halfspaces at $x \in \partial^* E$.



Reduced Boundary and Another Structure Theorem

Definition 3. [blowup]

$$E_r \equiv \{y \in \mathbb{R}^n | r(y-x) + x \in E\} \tag{12}$$



Theorem 4. [blowups are halfspaces] *For $x \in \partial^* E$,*

$$\chi_{E_r} \to \chi_{H^-(x)} \text{ in } L^1_{loc}(\mathbb{R}^n) \tag{13}$$

Reduced Boundary and Another Structure Theorem

Theorem 5. [structure theorem for sets of finite perimeter] *If E has locally finite perimeter then*

(1)

$$\partial^* E = \left(\bigcup_{k=1}^{\infty} K_k \right) \cup N, \tag{14}$$

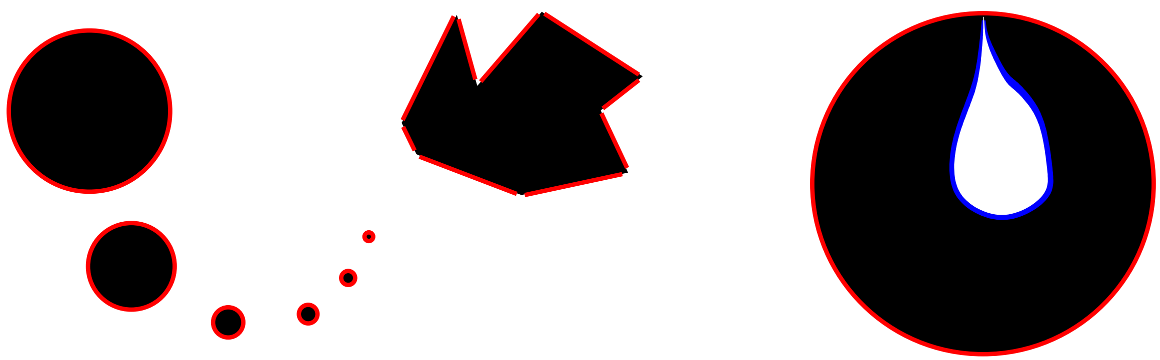
where

$$||\nabla \chi_E||(N) = 0 \tag{15}$$

and K_k is a compact subset of a C^1-hypersurface S_k $(k = 1, 2, ...)$,

(2) $\nu_E|_{S_k}$ is normal to S_k $(k = 1, 2, ...)$, and

(3) $||\nabla \chi_E|| = \mathcal{H}^{n-1} \llcorner \partial^* E$.



measure theoretic boundary and Guisti's boundary

How does the reduced boundary differ from the one that can be seen by integration – the *measure theoretic boundary*? How about other definitions of boundary?

Definition 4. [measure theoretic boundary] *We say that* $x \in \partial_* E$, *the* **measure theoretic boundary** *of* E, *if*

$$\limsup_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{r^n} > 0 \tag{16}$$

and

$$\limsup_{r \to 0} \frac{\mathcal{L}^n(B(x,r) - E)}{r^n} > 0. \tag{17}$$

Theorem 6. [size difference between $\partial^* E$ and $\partial_* E$] *If* E *has locally finite perimeter:*

(1) $\partial^* E \subset \partial_* E$.
(2) $\mathcal{H}^{n-1}(\partial_* E - \partial^* E) = 0$

measure theoretic boundary and Guisti's boundary

But there at least two other definitions of boundary we might consider.

Definition 5. [Giusti's boundary] *$x \in \partial_g E$ if*

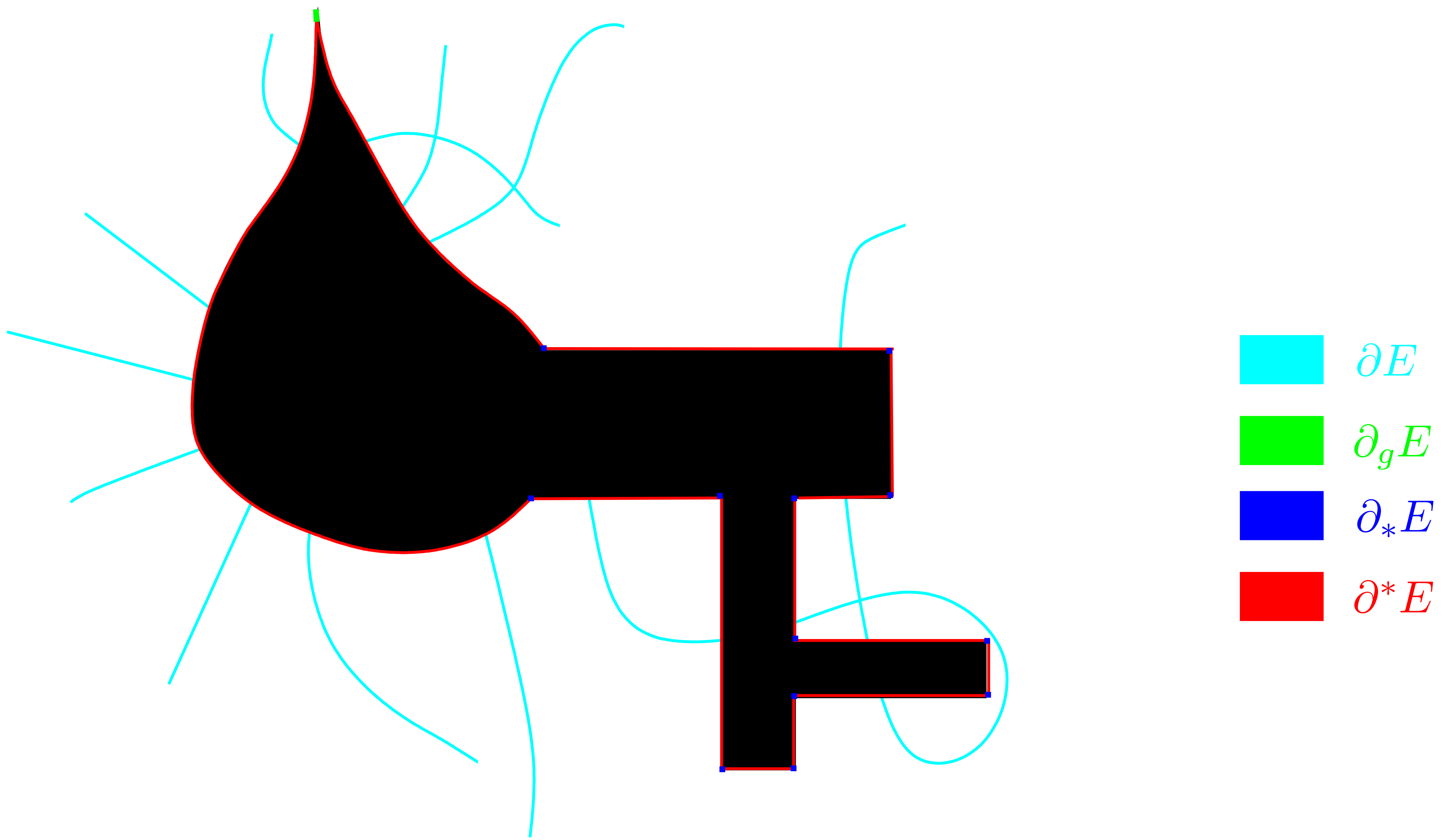
$$0 < |E \cap B(x,r)| < |B(x,r)| = \omega_n r^n \text{ for all } r > 0 \tag{18}$$

Definition 6. [topological boundary] *For $E \subset \mathbb{R}^n$: $x \in \partial E$, the topological boundary of E, if there exists points of E and E^c in $B(x,r)$ for all $r > 0$.*

The relationship is simple: $\partial^* E \subset \partial_* E \subset \partial_g E \subset \partial E$

Relations Between the Different Boundaries

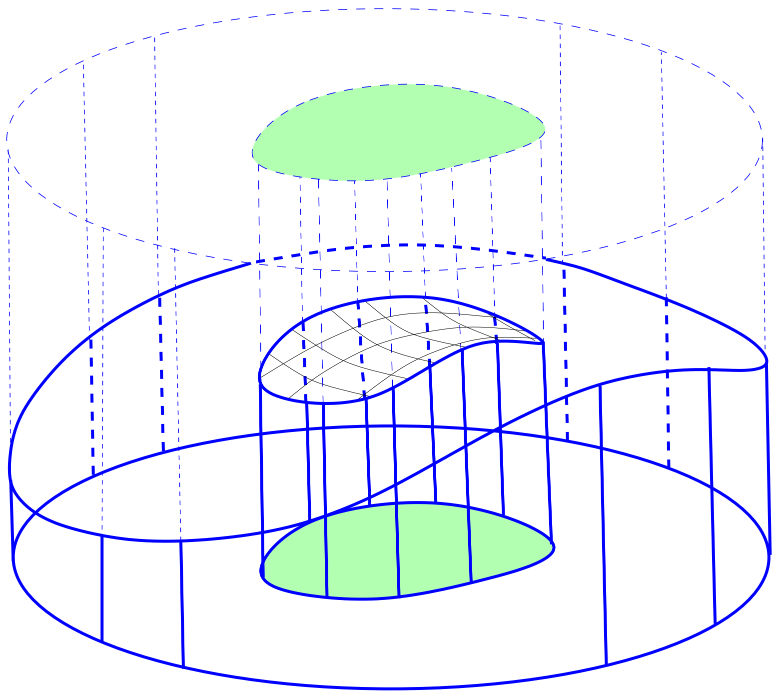
A picture illustrating: $\partial^* E \subset \partial_* E \subset \partial_g E \subset \partial E$



Finally: a regularity result for minimal surfaces!

Theorem 7. *Suppose that E is locally minimal. Then*

(1) $\partial_g E$ is analytic in a neighborhood of every point in $\partial^* E$ and
(2) $H^{n-1}(\partial_g E - \partial^* E) = 0$.

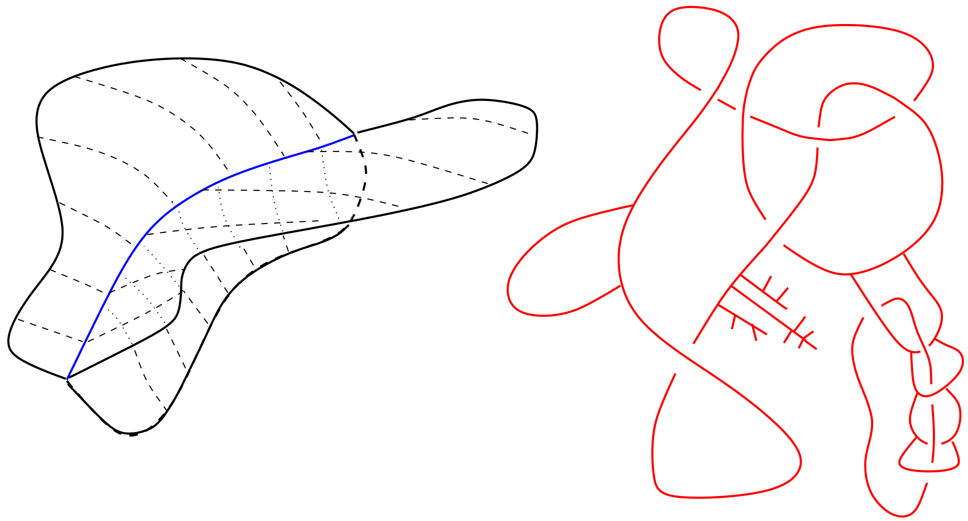


Another approach: Currents

What are currents?

A k-dimensional *integer multiplicity rectifiable current* in $\mathbb{R}^n$ is an element of the dual space to k-forms in $\mathbb{R}^n$ which can be represented by integration against an oriented k-rectifiable subset with positive integer density. A picture:

Why use currents? Answer: triple junctions and higher codimension.



This was the path to minimal surfaces introduced by Federer and Fleming in 1960. But this is a subject for another talk ...

A look ahead to the fifth and final lecture

* Geometry in high dimensions: examples from spheres
* Concentration of measure: fundamental inequality
* Consequences of concentration of measure
* Projections (almost) preserving distances: Johnson-Lindenstrauss
* Application of Johnson-Lindenstrauss: Indexing and Clustering

Lecture 5: High dimensional Geometry

As promised, we now look at some fascinating aspects of geometry in high dimensions.

A distinction between finite and infinite dimensions is typical. I will instead dwell on properties of high, but finite dimension.

Inspiration: David Donoho's talk at Browder's 2000 UCLA shindig.

Inspiration: Dimension reduction is Universal: "Everything" is high-dimensional, yet humans seem to be able to extract useful low dimensional models and use these effectively for various problem solving.

I will use the board a great deal during this lecture: the slides are only very(!) cryptic notes to be filled out on the board. I plan to add notes so that the posted version will stand alone better.

Examples: Balls and Spheres in High Dimensions

We start by deriving answers to questions about balls and spheres in $\mathbb{R}^n$. We will be most intereted in understanding what happens as n gets large.

Compute: n-dimensional Volume of ball in $\mathbb{R}^n$

Compute: (n-1)-dimensional volume of sphere in $\mathbb{R}^n$

Q1: At what radius does the ball have unit volume?

Q2: At what radius does the sphere have unit n-1 volume?

Q3: What is the relationship between Vol(S^n) and Vol(B^n)?

Q4: At what radius r is Vol$(B^n(r)) = \epsilon$Vol$(B^n(1))$?

Q5: How can we get the volume of a sphere from the volume of the ball?

Concentration of Measure

We have already seen that $\frac{N-1}{N}$ of the ball's volume is contained in the outer $\epsilon = \frac{\ln N}{n}$ shell of the unit ball, that the $\sqrt{2\epsilon}$ neighborhood of any equator (great circle) contains at least $\frac{N-1}{N}$ of the volume of the sphere. This phenomena is called the concentration of measure ($=$ volume).

Now we get more general concentration inequalities via isoperimetric inequalities.

(1) Isoperimetric inequality on spheres

(2) Main result: concentration on the sphere

(i) $A, B \in S^n$, B is geodesic ball, $\mu(A) = \mu(B)$ then

$$\mu(A_r) \geq \mu(B_r) \forall r > 0 \tag{19}$$

(ii) $B \subset S^n$ a geodesic ball, $\mu(B) \geq \frac{1}{2}$ then

$$1 - \mu(B_r) \leq \mathrm{e}^{-\frac{(n-1)r^2}{2}} \quad r > 0 \tag{20}$$

(iii) Result: if $\mu(A_r) \geq \frac{1}{2}$ then

$$1 - \mu(A_r) \leq \mathrm{e}^{-\frac{(n-1)r^2}{2}} \quad r > 0 \tag{21}$$

(3) Lipschitz functions on spheres: "almost" constant "almost" everywhere on S^n.

Dimension Reduction: Johnson-Lindenstrauss Lemma

Lemma 1. [Johnson-Lindenstruass] *For any* $0 < \epsilon < 1$ *and any integer* n*, choose*

$$k \geq \frac{4 \ln n}{\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3}}. \tag{22}$$

Then for any set V *of* n *points in* $\mathbb{R}^n$*, there is a projection* f *such that*

$$(1-\epsilon)||u-v||^2 \leq ||f(u)-f(v)|| \leq (1+\epsilon)||u-v||^2 \tag{23}$$

for all $u, v \in V$*.*

Lemma 2. *(Vempala, lemma 1.3) Let each entry of the* $n \times k$ *matrix* R *be chosen independently from* $N(0,1)$*. Let* $v = \frac{1}{\sqrt{k}}\mathrm{R}^T u$ *for* $u \in \mathbb{R}^n$*. Then for any* $\epsilon > 0$*,*

(1) $\mathrm{E}(||v||^2) = ||u||^2$

(2) $\mathrm{P}(|||v||^2 - ||u||^2| \geq \epsilon||u||^2) < 2\mathbf{e}^{-(\epsilon^2-\epsilon^3)\frac{k}{4}}$

Q1 What kinds of reductions can we get using Johnson-Lindenstrauss?

Q2 How hard is it to find a good projection?

Johnson-Lindenstrauss Applied: Indexing and Clustering

Applications such as *Latent Semantic Indexing* have met with considerable success in applications such as search engines. A matrix that represents a large number of documents is reduced in complexity through the use of the SVD to compute a low rank approximation.

Q1 How can the SVD be used to get a low rank approximation?

Q2 How can dimension reducing projections help reduce the computation cost in obtaining a low rank approximation?

Q3 What is the savings?

Summary of Part Two: Lectures 4 – 5

There is much more to say of course, but I will stop here. After all:

The secret to wearying consists in saying everything

Voltaire

Geometric Analysis: is both useful for real applications and deeply interesting in it's own right. I have presented a few peaks at more advanced theory in order to inspire a more careful study.

High-dimensional Geometry: we live surrounded by data and physical reality that is well approximated by high-dimensional models. Yet, our minds are capable of processing and modeling the information efficiently. This suggests dimension reduction and other simplifying approximations. Exploiting the peculiarities of high dimensions can help us a great deal with real problems.

References for part II

0 I define geometric analysis more broadly than is typical: for me it refers to the geometry of sets, functions and measures in Euclidean spaces or manifolds, with an emphasis on the connections to, and use of, analysis as opposed to a special emphasis on topology or algebra. Accordingly, the term refers to an organically contiguous body of knowledge and connected research threads that includes geometric measure theory and large chunks of PDEs, harmonic analysis, variational analysis (including nonsmooth analysis), differential geometry and nonlinear functional analysis.

1 I used both Evans and Gariepy's (1999) "Measure Theory and Fine Properties of Functions" and Enrico Giusti's (1984) "Minimal Surfaces and functions of Bounded Variation" as my main references when preparing lecture 4. A more complete list of references useful when digging into geometric analysis include:

1 "Measure Theory and Fine Properties of Functions" (1999) Lawrence C. Evans and Ronald F. Gariepy

2 "Minimal Surfaces and functions of Bounded Variation" (1984) Enrico Giusti

3 "Seminar on Geometric Measure Theory" (1986), Robert Hardt and Leon Simon

4 "Lectures on Geometric Measure Theory" (1983), Leon Simon

5 "Geometric Measure Theory: A Beginner's Guide" (2000), Frank Morgan

6 "Geometric Measure Theory" (1969), Herbert Federer

7 "Geometric Measure Theory – an Introduction" (2002), Fanghua Lin and Xiaoping Yang

8 "Plateau's Problem: An Invitation to Varifold Geometry" (1966: Revised Edition,2001), Fred

References for part II

Almgren

9 "Selected Works of Frederick J. Almgren, Jr." (1999) edited by Jean Taylor

10 "Geometry of Sets and Measures in Euclidean Spaces" (1995), Pertti Mattila

11 "Singular Integrals and Differentiability Properties of Functions" (1970), Elias M. Stein

11 For Background in Riemannian Geometry: "Riemannian Geometry" (1992), Manfredo Perdigao Do Carmo

12 For Background in PDE: "Partial differential Equations" (1998), L. Craig Evans

13 For Background in Harmonic Analysis: "Fourier Analysis" (2001) Javier Duoandikoetxea

2 Other monographs of speciallized interest include: "Klaus Ecker's Regularity theory for Mean Curvature Flow", Tom Ilmanen's "Elliptic Regularization and Partial Regularity for Motion by Mean Curvature", Leon Simon's "Theorems on regularity and Singularity of Energy Minimizing Maps", Mariano Giaquinta's "Introduction to Regularity theory for Nonlinear Elliptic Systems", David and Semmes' "Analysis of and on Unifromly rectifiable Sets", and last but definitely not least, Ambrosio, Fusco and Pallara's "Functions of Bounded Variation and Free Discontinuity Problems".

3 Papers: the papers in the selected works of Fred Almgren'a refered to above, Allard's papers from the 70's on Varifolds, and then papers refered to in the texts and monographs referenced to above. In particular, I recommend the bibliographies from Frank Morgan's book, the book by

References for part II

Fanghua Lin and Xiaoping Yang, and Mattila's book.

4 Main references for lecture 5 were "The Concentration of Measure Phenomenon" (2001) by Michel Ledoux and "The Random Projection Method" (2004) by Santosh S. Vempala. I also used the paper by Sanjoy Dasgopta and Anupam Gupta, "An Elementary Proof of the Johnson-Lindenstrauss Lemma" TR-99-006, International Computer Science Institute.

Notes for part II

1: Page 30 : The sets for which the isoperimetric inequality applies
2: Page :
3: Page :
4: Page :
5: Page :
6: Page :
7: Page :